Adinkras as Origami

Arsh Chhabra (Pomona College), Xuehuai He (Pomona College),
Elena O'Grady (Reed College), Melinda Yang (Pomona College)

Last Updated: August 28, 2023

Outline

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors, and do not necessarily reflect the views of the National Science Foundation.

Motivation

History of Adinkras

Figure: Michael Faux and Sylvester "Jim" Gates

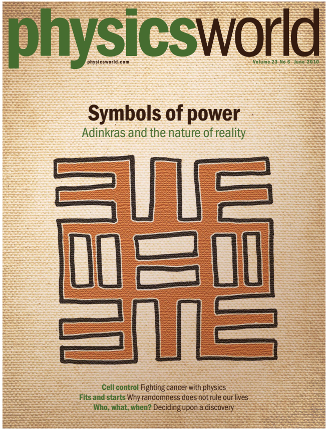


Figure: June 2010 Cover of *Physics World*

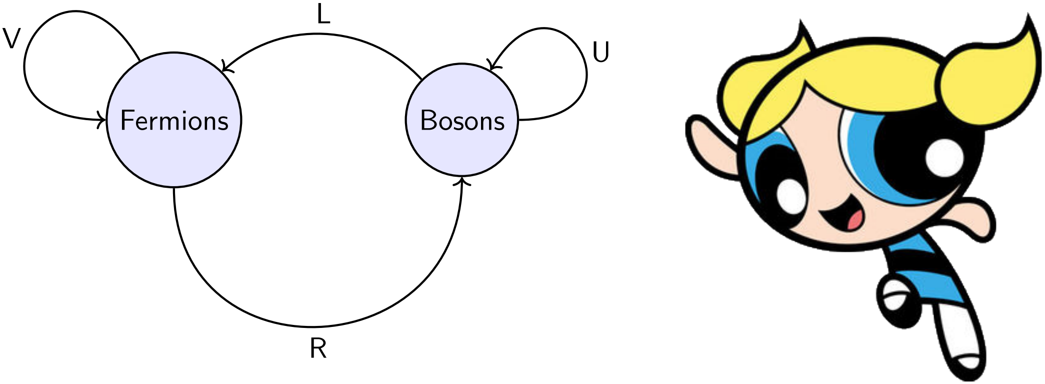


Figure: The PPG Diagram and "Bubbles" from the Power Puff Girls

Codes and heights

- $\mathbb{F}_2 = \{0, 1\}$.
- Vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{F}_2^n$ are called **codes**.
- Height of the codes $\mathrm{ht} : \mathbb{F}_2^n \mapsto \mathbb{Z}$ as the number of components of $\mathbf{v}$ with $v_i = 1$.
- A code is **even** if $\mathrm{ht}(\mathbf{v}) \in 2\mathbb{Z}$.
- A code is **doubly even** if $\mathrm{ht}(\mathbf{v}) \in 4\mathbb{Z}$.

Examples

- $\mathbf{v} = (1, 1) \in \mathbb{F}_2^2$
  $\mathrm{ht}(\mathbf{v}) = 2$
- $\mathbf{v} = (1, 0, 1) \in \mathbb{F}_2^3$
  $\mathrm{ht}(\mathbf{v}) = 2$
- $\mathbf{v} = (0, 0, 0, 1) \in \mathbb{F}_2^4$
  $\mathrm{ht}(\mathbf{v}) = 1$
- $\mathbf{v} = (1, 1, 1, 1) \in \mathbb{F}_2^4$
  $\mathrm{ht}(\mathbf{v}) = 4$

Pomona College

- Choose a subspace $C \subseteq \mathrm{ht}^{-1}(4\mathbb{Z})$ consisting of **doubly even codes**.
- Draw a bipartite graph with:
  - "Black" vertices:
    $B = \mathrm{ht}^{-1}(2\mathbb{Z})/C$;
  - "White" vertices:
    $W = \mathrm{ht}^{-1}(2\mathbb{Z}+1)/C$;
  - Edges:
    $E = \{(v,w) \in \mathbb{F}_2^n \times \mathbb{F}_2^n : \mathrm{ht}(v-w) = 1\}/C.$

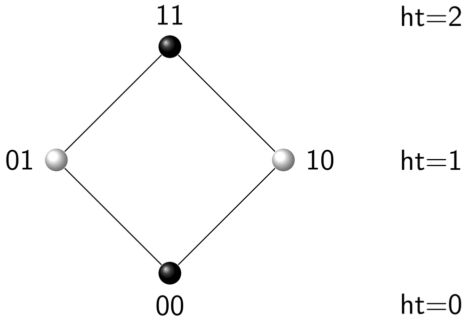
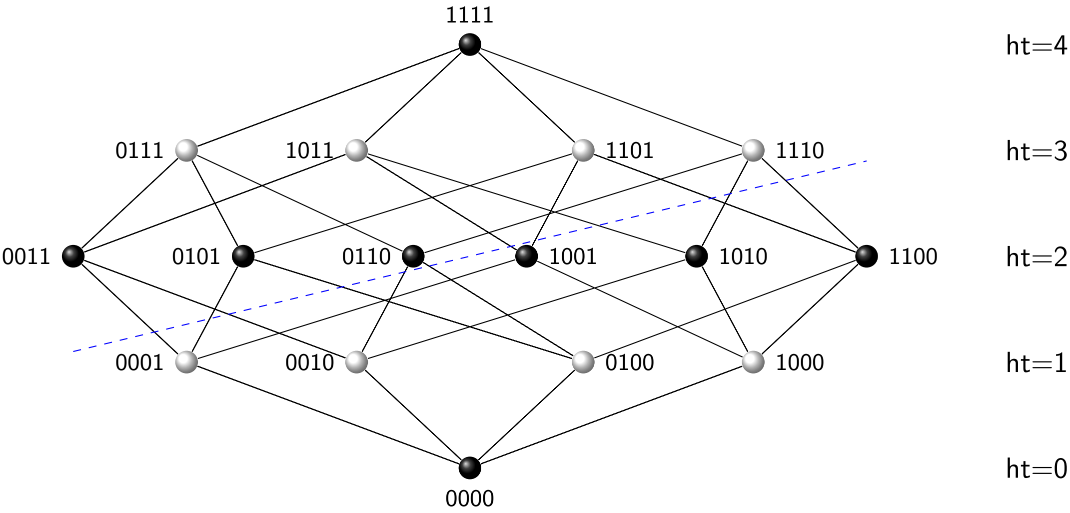
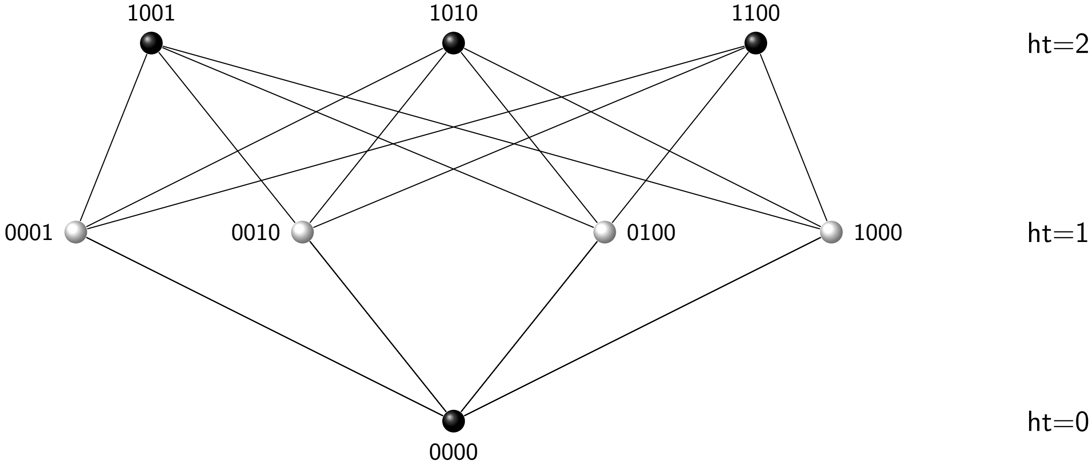


Figure: $n = 2$ Adinkra



$n = 4, C = \{0000\}$ Adinkra



$$n = 4, C = \{0000, 1111\}$$

Can we find different ways to generate Adinkras?

- Every **compact, connected Riemann surface** S can be defined by a single polynomial

$$f(x,y) = \sum_{i,j} a_{ij} x^i y^j .$$

- Let $\beta : S \to \mathbb{P}^1(\mathbb{C})$ be a rational function. A **critical value** for β is a complex number $q = \beta(P)$ for some point $P = (x_0, y_0)$ which satisfies

$$f(P) = 0 \qquad \text{and} \qquad \frac{\partial \beta}{\partial x}(P)\,\frac{\partial f}{\partial y}(P) - \frac{\partial \beta}{\partial y}(P)\,\frac{\partial f}{\partial x}(P) = 0.$$

- A **Belyĭ pair** (S, β) is the surface S together with a rational function $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which has critical values $q \in \{0,\ 1,\ \infty\}$.

Let $\mathbb{P}^1(\mathbb{C}) = \left\{(x, y) \in \mathbb{C}^2 \colon y = 0\right\} \cup \{\infty\}$.

We define **stereographic projection** by the map

$$\mathbb{P}^1(\mathbb{C}) \to S^2(\mathbb{R}),$$

$$(x, y) \mapsto \left(\frac{2\,\mathrm{Re}(x)}{|x|^2+1}, \frac{2\,\mathrm{Im}(x)}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1}\right).$$

As such, we call $\mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$ the **Riemann Sphere**.

Figure: Unit Sphere $S^2(\mathbb{R})$

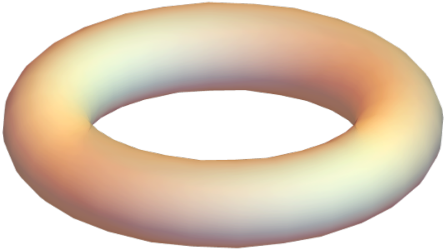


Figure: Torus $\mathbb{T}^2(\mathbb{R})$

An **elliptic curve** E is an equation of the form $y^2 = x^3 + A\,x + B$ with A and B complex numbers such that $4\,A^3 + 27\,B^2 \neq 0$.

There is an **elliptic logarithm** which induces a map

$$E(\mathbb{C}) = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}_E\};$$

$$\downarrow$$

$$\mathbb{T}^2(\mathbb{R}) \simeq (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$$

Hence, the set of complex points on an elliptic curve is a Riemann surface of genus 1.

Example: Riemann sphere

Recall $S = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. For any natural number n, define the rational map $\beta\colon S \to \mathbb{P}^1(\mathbb{C})$ via

$$\beta(z) = \frac{z^n}{z^n + 1}.$$

This is a Belyĭ map on the Riemann Sphere.

Example: Torus

Consider the elliptic curve $E\colon y^2 = x^3 - x$. Recall $S = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ is the torus. Define the rational map $\beta\colon S \to \mathbb{P}^1(\mathbb{C})$ via:

$$\beta(x, y) = \frac{(x^4 - 6\,x^2 + 1)^4}{(x^8 + 20\,x^6 - 26\,x^4 + 20\,x^2 + 1)^2}.$$

This is a Belyĭ map on the torus.

Fix a Belyĭ pair (S, β). We construct a bipartite graph on S as follows:

- $B = \beta^{-1}(0)$ corresponds to "black" vertices.
- $W = \beta^{-1}(1)$ corresponds to "white" vertices.
- $F = \beta^{-1}(\infty)$ corresponds to centers of faces.
- $E = \beta^{-1}([0,1])$ corresponds to edges.

We call this graph a **Dessin d'Enfant**.

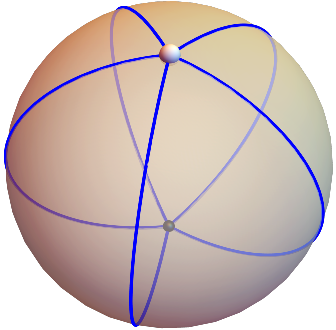


Figure: The Dessin d'Enfant for $\beta(z) = \dfrac{z^5}{z^5+1}$

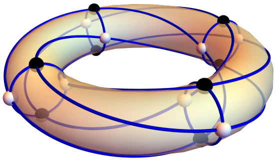


Figure: The Dessin d'Enfant corresponding to β.

Let $S = E(\mathbb{C})$ be the set of complex points on the elliptic curve $E\colon y^2 = x^3 - x$. Let $\beta\colon S \to \mathbb{P}^1(\mathbb{C})$ be the Toroidal Belyǐ map given by:

$$\beta(x, y) = \frac{(x^4 - 6\,x^2 + 1)^4}{(x^8 + 20\,x^6 - 26\,x^4 + 20\,x^2 + 1)^2}.$$

The corresponding Dessin d'Enfant has

- $|B| = 8$ "black" vertices,
- $|W| = 8$ "white" vertices,
- $|F| = 16$ faces, and
- $|E| = 32$ edges.

Proposition (Doran, Iga, Kostiuk, Landweber, Méndez-Diez; 2015)

Fix an integer $n \geq 2$. Let ζ be a primitive $2n$th root of unity, and denote $\sigma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ as that Möbius transformation satisfying $\sigma(\zeta) = 0$, $\sigma(\zeta^3) = 1$, and $\sigma(\zeta^{2n-1}) = \infty$.

(a) Define the set

$$S = \left\{ (x_1 : x_2 : \cdots : x_n) \in \mathbb{P}^{n-1}(\mathbb{C}) \;\middle|\; \begin{array}{c} \sigma(\zeta^{2k-1})\, x_1^2 + x_2^2 + x_{k+1}^2 = 0 \\ \text{for } k = 2, 3, \ldots, n-1 \end{array} \right\}.$$

Then S is a compact, connected Riemann surface with genus $g(S) = 1 + 2^{n-3} \cdot (n-4)$.

(b) There exists a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which sends

$$P = (x_1 : \cdots : x_n) \quad \mapsto \quad z = \sigma^{-1}\left(-\frac{x_2^2}{x_1^2}\right) \quad \mapsto \quad \frac{z^n}{z^n + 1}.$$

Its Dessin d'Enfant has $|B| = 2^{n-1}$ "black" vertices, $|W| = 2^{n-1}$ "white" vertices, $|E| = 2^{n-1} \cdot n$ edges, and $|F| = 2^{n-2} \cdot n$ rectangular faces.

(c) Every Adinkra can be constructed using the Belyĭ pair (S, β).

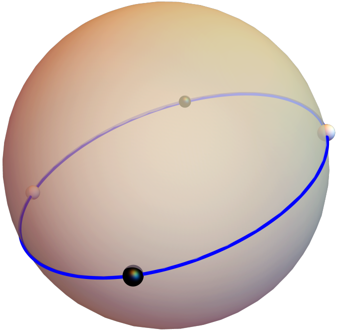


Figure: Adinkra as a Dessin d'Enfant

$$
\begin{array}{ccc}
S = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R}) & & P = (x, y) \\
\downarrow \varphi & & \downarrow \\
B_2 = \mathbb{P}^1(\mathbb{C}) & & z = i\,\dfrac{x^2 - 1}{x^2 + 1} \\
\downarrow \widetilde{\beta} & & \downarrow \\
\mathbb{P}^1(\mathbb{C}) & & q = \dfrac{z^2}{z^2 + 1} = -\dfrac{(x^2 - 1)^2}{4\,x^2}
\end{array}
$$

Example: Adinkra for $n = 3$ and $C = \{000\}$

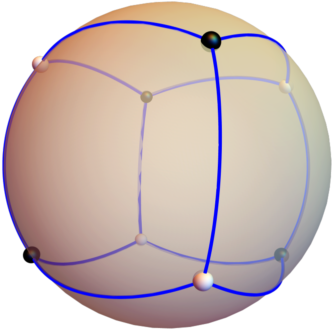


Figure: Adinkra as a Dessin d'Enfant

$$S = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R}) \qquad P = (x, y)$$

$$\downarrow \varphi \qquad \qquad \downarrow$$

$$B_3 = \mathbb{P}^1(\mathbb{C}) \qquad z = \frac{x^4 - 2\sqrt{2}\,x}{2\sqrt{2}\,x^3 + 1}$$

$$\downarrow \widetilde{\beta} \qquad \qquad \downarrow$$

$$\mathbb{P}^1(\mathbb{C}) \qquad q = \frac{z^3}{z^3 + 1} = \frac{x^3\left(x^3 - 2\sqrt{2}\right)^3}{\left(x^6 + 5\sqrt{2}\,x^3 - 1\right)^2}$$

Example: Adinkra for $n = 4$ and $C = \{0000, 1111\}$

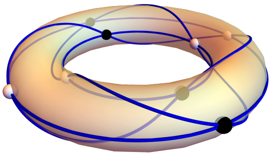


Figure: Adinkra as a Dessin d'Enfant on $E : y^2 = x^3 - x$

$$S = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R}) \qquad P = (x, y)$$

$$\downarrow \varphi \qquad\qquad \downarrow$$

$$B_4 = \mathbb{P}^1(\mathbb{C}) \qquad z = \frac{1+i}{\sqrt{2}} \, \frac{x^2+1}{2\,y}$$

$$\downarrow \widetilde{\beta} \qquad\qquad \downarrow$$

$$\mathbb{P}^1(\mathbb{C}) \qquad q = \frac{z^4}{z^4+1} = \frac{(x^2+1)^4}{(x^4 - 6\,x^2 + 1)^2}$$

Example: Adinkra for $n = 4$ and $C = \{0000\}$

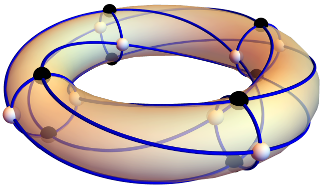


Figure: Adinkra as a Dessin d'Enfant on $E : y^2 = x^3 - x$

$$\begin{array}{ccc}
S = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R}) & & P = (x, y) \\
\downarrow \varphi & & \downarrow \\
B_4 = \mathbb{P}^1(\mathbb{C}) & & z = \dfrac{i}{2\sqrt{2}} \, \dfrac{x^4 - 6\,x^2 + 1}{(x^2 + 1)\, y} \\
\downarrow \widetilde{\beta} & & \downarrow \\
\mathbb{P}^1(\mathbb{C}) & & q = \dfrac{(x^4 - 6\,x^2 + 1)^4}{(x^8 + 20\,x^6 - 26\,x^4 + 20\,x^2 + 1)^2}
\end{array}$$

Pomona College

What else do we know about the Belyĭ pair (S, β)?

- Doran et al. factor the Belyĭ map $\beta = \widetilde{\beta} \circ \varphi$ through a map $\widetilde{\beta}\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ on the sphere to focus on the coloring of the edges.
- Can we factor $\beta = \eta \circ \phi$ through a map $\eta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ on the torus to focus on the rectangular faces?

Let S be a compact, connected Riemann Surface defined as

$$S\colon \left\{ (x_1 : x_2 : \cdots : x_n) \in \mathbb{P}^{n-1}(\mathbb{C}) \;\middle|\; \begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 0 \\ \sigma(\zeta^5)x_1^2 + x_2^2 + x_4^2 &= 0 \\ &\vdots \\ \sigma(\zeta^{2k-1})x_1^2 + x_2^2 + x_{k+1}^2 &= 0 \\ &\vdots \\ \sigma(\zeta^{2n-3})x_1^2 + x_2^2 + x_n^2 &= 0 \end{aligned} \right\}.$$

Fix two integers r and s satisfying $1 < r < s < n$. We define the **quadric intersection**

$$E(\mathbb{C}) = \left\{ (x_1 : x_2 : x_{r+1} : x_{s+1}) \in \mathbb{P}^3(\mathbb{C}) \;\middle|\; \begin{aligned} \sigma(\zeta^{2r-1})\, x_1^2 + x_2^2 + x_{r+1}^2 &= 0 \\ \sigma(\zeta^{2s-1})\, x_1^2 + x_2^2 + x_{s+1}^2 &= 0 \end{aligned} \right\}.$$

PRiME 2023 Theorem 1.1

For integers r and s satisfying $1 < r < s < n$, the quadric intersection

$$E(\mathbb{C}) = \left\{ (x_1 : x_2 : x_{r+1} : x_{s+1}) \in \mathbb{P}^3(\mathbb{C}) \,\middle|\, \begin{matrix} \sigma(\zeta^{2r-1})\, x_1^2 + x_2^2 + x_{r+1}^2 = 0 \\ \sigma(\zeta^{2s-1})\, x_1^2 + x_2^2 + x_{s+1}^2 = 0 \end{matrix} \right\}$$

is an elliptic curve which has j-invariant

$$j(E) = 256\,\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2\,(\lambda - 1)^2} \quad \text{in terms of} \quad \lambda = \frac{\sigma(\zeta^{2r-1})}{\sigma(\zeta^{2r-1}) - \sigma(\zeta^{2s-1})}.$$

PRiME 2023 Theorem 1.2

Consider the Belyĭ pair (S, β) as in Doran et al.
We factor the Belyĭ map $\beta = \eta \circ \phi$ in terms of that Toroidal Belyĭ map η which sends $Q = (x, y)$ in $E(\mathbb{C})$ to $q = z^n/(z^n + 1)$ in $\mathbb{P}^1(\mathbb{C})$ in terms of

$$z = \frac{(x^2 - 2\,x + \lambda)^2 - \zeta\,\tau\,(x^2 - \lambda)^2}{\zeta\,(x^2 - 2\,x + \lambda)^2 - \tau\,(x^2 - \lambda)^2} \quad \text{where} \quad \tau = \sin\frac{q\pi}{n} \bigg/ \sin\frac{(q-1)\pi}{n}.$$

Given a map $\phi \colon S \to T$ between compact, connected Riemann surfaces, denote $e_\phi(P)$ to be the **ramification index** of ϕ at P, which is an integer that effectively measures how much ϕ fails to be a covering map at P.

- A **critical point** is a point $P \in S$ with $e_\phi(P) > 1$.
- The correpsonding $Q = \phi(P)$ is called a **critical value**.

Definition

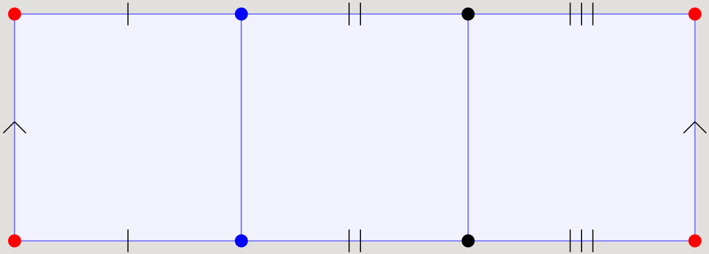
Let $E : y^2 = x^3 + A\,x + B$ be an elliptic curve; recall that $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ which can be represented by a rectangle. A nonconstant morphism $\phi : S \to E(\mathbb{C})$ whose critical values $Q \in \{\mathcal{O}_E\}$ is said to be an **origami**. Its degree is the integer

$$N = \sum_{P \in V} e_\phi(P) = |V| + (2\,g(S) - 2) \qquad \text{where} \qquad V = \phi^{-1}(\mathcal{O}_E).$$

Pomona College

Example

Let E, E' be elliptic curves and $\phi\colon S = E(\mathbb{C}) \to E'(\mathbb{C})$ be an origami with $\deg\phi = 3$. Recall that $E(\mathbb{C})$ is a Riemann surface of genus 1, and $E'(\mathbb{C})$ can be represented by a rectangle, so we can then tile the torus with three squares as follows:



Each colored vertex then corresponds to a unique point in $V = \phi^{-1}(\mathcal{O}_E)$.

PRiME 2023 Theorem 2

Consider the Belyĭ pair (S, β) as in Doran et al. Assume that $\beta = \eta \circ \phi$ for some nonconstant maps $\eta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and $\phi : S \to E(\mathbb{C})$.

1. η must be a Toroidal Belyĭ map.
2. ϕ cannot be an origami whenever $n \geq 6$.

Pomona College

PRiME 2023 Theorem 2.1

Consider the Belyĭ pair (S, β) as in Doran et al. Assume that $\beta = \eta \circ \phi$ for some nonconstant maps $\eta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and $\phi : S \to E(\mathbb{C})$. Then η must be a Toroidal Belyĭ map.

Sketch of Proof:

- We know that $\beta = \eta \circ \phi$
- $e_\beta(P) = e_\phi(P)\, e_\eta(\phi(P))$ for all points $P \in S$
- If $e_\eta(\phi(P)) \neq 1$ for some $\phi(P) \in E(\mathbb{C})$, then $e_\beta(P) \neq 1$
- $e_\beta(P) \neq 1$ only when $\beta(P) \in \{0, 1, \infty\}$
- So $\eta(\phi(P)) = \beta(P) \in \{0, 1, \infty\}$, making η a Belyĭ map $\square$

PRiME 2023 Theorem 2.2

Consider the Belyĭ pair (S, β) as in Doran et al. Assume that $\beta = \eta \circ \phi$ for some nonconstant maps $\eta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and $\phi : S \to E(\mathbb{C})$. Then ϕ cannot be an origami whenever $n \geq 6$.

Sketch of Proof.

- Partition the set $\phi^{-1}(\mathcal{O}_E)$ by $e_\phi(P)$ and $\beta(P)$; deduce that exactly one subset is nonempty.
- Assume $\eta(\mathcal{O}_\mathcal{E}) = 0$ or 1.
- Use the Hurwitz genus formula to come up with a lower bound on $j = e_\phi(P)$ for $P \in \phi^{-1}(\mathcal{O}_E)$, and use the multiplicativity of ramification indices to come up with an upper bound.
- Consider the possible cases for j, based on the bounds, finding contradictions using the fact that $\deg \phi = \sum_{P \in \phi^{-1}(\mathcal{O}_E)} e_\phi(P)$.
- Find similar contradictions in the case where $\eta(\mathcal{O}_E) = \infty$.


Impostor
ORIGAMI
imgflip.com

Future Work

- Adinkras are constructed from subspaces $C \subseteq \mathbb{F}_2^n$; they are quotients of the hypercube. We know that they can be embedded on a compact, connected Riemann surface of genus $g(S) = 1 + 2^{n-m-3} \cdot (n-4)$. Find explicit embeddings when $n \geq 5$.
- The Belyĭ map $\eta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ in Theorem 1 has degree $\deg \eta = 8\,n$. Factor $\eta = \lambda \circ \gamma$ for (a) some $\gamma : E(\mathbb{C}) \to E'(\mathbb{C})$ with $\deg \gamma = 8$ and (b) some Toroidal Belyĭ map $\lambda : E'(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of $\deg \lambda = n$ whose Dessin d'Enfant has exactly one "black" vertex and one "white" vertex.

Selected References

Charles Doran, Kevin Iga, Jordan Kostiuk, Greg Landweber, and Stefan Méndez-Diez, *Geometrization of N-extended 1-dimensional supersymmetry algebras, I*, Adv. Theor. Math. Phys. **19** (2015), no. 5, 1043–1113. MR 3487651

Ernesto Girondo and Gabino González-Diez, *Introduction to compact Riemann surfaces and dessins d'enfants*, London Mathematical Society Student Texts, vol. 79, Cambridge University Press, Cambridge, 2012. MR 2895884

Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR 2514094

Acknowledgements

We would like to thank:

- Department of Mathematics and Statistics at Pomona College;
- Summer Undergraduate Research Program at Pomona College;
- National Science Foundation (DMS-2113782);
- Professors Alex Barrios, Luis Puente Garcia, Haydee Lindo, and Lori Watson; and
- Mark Curiel, Olivia Del Guercio, Fabian Ramirez, and Japheth Varlack; and
- Cameron Thomas and Professor Edray Goins for all of their guidance.

Thank You!

Questions?